TRINITY COLLEGE

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3

Section One: Calculator-free

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words SOLUTIONS

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	7	7	50	53	35
Section Two: Calculator-assumed	12	12	100	98	65
			Total	151	100

Instructions to candidates

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2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free

35% (53 Marks)

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 (6 marks)

Alternatively divide this by z^2-z-2, then use quadratic formula.

Consider $f(z)=z^4+3z^3+7z^2-21z-26$, $z \in \mathbb{C}$. Solve $f(z)=0$ over $\mathbb{C}$, given $f(-1)=f(2)=0$

$$\Rightarrow f(z) = (z+1)(z-2)(z^2+az+b)$$
$$= (z^2-z-2)(z^2+az+b)$$

$\Rightarrow -2b = -26$ ✓ Equating constant terms

$\therefore \underline{b = 13}$

$\therefore f(z) = (z^2-z-2)(z^2+az+13)$ ✓

$\Rightarrow -2-a+13 = 7$ ✓ Equating coefficients of z^2

$\therefore \underline{a=4}$

Let: $f(z) = (z+1)(z-2)(z^2+4z+13) = 0$

Consider: $z^2+4z+4-4+13=0$ ✓

$\Rightarrow (z+2)^2+9=0$

$\Rightarrow (z+2)^2 = -9$

$\therefore z+2 = \pm 3i$ ✓

$\therefore$ the solutions to $f(z)=0$ over $\mathbb{C}$ are:

$\underline{z = -1, 2, -2+3i, -2-3i}$ ✓

Question 2 **(7 marks)**

A sphere has equation $x^2 + y^2 + z^2 - 2x + 4y + 3z + 1 = 0$.

(a) Determine the coordinates of the centre and the radius of the sphere. (4 marks)

$$\Rightarrow \quad x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 + 3z + \frac{9}{4} = 1 + 4 + \frac{9}{4} - 1 \quad \checkmark$$

$$\Rightarrow \quad (x-1)^2 + (y+2)^2 + \left(z + \frac{3}{2}\right)^2 = \frac{25}{4} \quad \checkmark$$

$\therefore$ Centre is $\underline{\left(1, -2, -\frac{3}{2}\right)}$ $\checkmark$

$$\text{Radius} = \sqrt{\frac{25}{4}}$$

$$= \underline{\underline{\frac{5}{2}}} \quad \checkmark$$

(b) Determine the vector equation of the straight line that passes through the points on the sphere where $y = -2$ and $z = 0$. (3 marks)

When $y = -2$ and $z = 0$:

$$x^2 + 4 - 2x - 8 + 1 = 0 \quad \checkmark$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x-3)(x+1) = 0$$

$\therefore \underline{x = 3}$ or $\underline{x = -1}$ $\checkmark$

Using the points: $(3, -2, 0)$ and $(-1, -2, 0)$

$$\underset{\sim}{r} = (3, -2, 0) + \lambda(-1-3, 0, 0)$$

$$= \underline{(3 - 4\lambda)\underset{\sim}{i} - 2\underset{\sim}{j}} \quad \checkmark$$

or similar/equivalent.

using:

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$$

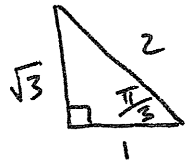
Question 3 **(8 marks)**

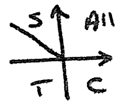
(a) Let $z = 2\cos\left(\frac{2\pi}{3}\right) + 2i\sin\left(\frac{2\pi}{3}\right)$.

(i) Express z in Cartesian form. (2 marks)

$$z = 2\left(\frac{-1}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right)i$$

$$= \underline{\underline{-1 + \sqrt{3}\,i}}$$





(ii) Determine z^5 in Cartesian form. (3 marks)

$$z^5 = \left(2\text{ cis }\frac{2\pi}{3}\right)^5$$

$$= 32\text{ cis}\left(\frac{10\pi}{3}\right)$$

$$= 16 \times 2\text{cis}\left(\frac{-2\pi}{3}\right)$$

$$= 16 \times \bar{z}$$

$$= 16\left(-1 - \sqrt{3}\,i\right)$$

$$= \underline{\underline{-16 - 16\sqrt{3}\,i}}$$

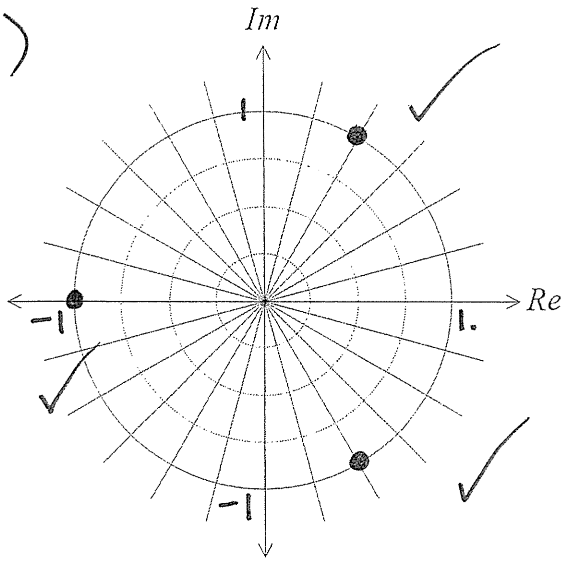
Notice: $\frac{10\pi}{3} = \frac{-2\pi}{3}$

Recall:
If $z = r\text{cis}\theta$
then $\bar{z} = r\text{ cis}(-\theta)$

Comment/Hint:
When $\pm$ $\mathbb{C}$ easier in Cartesian Form ie. Componentwise
When $\times \div$ $\mathbb{C}$ easier in Polar (cis) Form.

(b) If $w^3 + 1 = 0$, sketch the location of all roots of this equation on the axes below. (3 marks)

$\Rightarrow w^3 = -1$
(Roots of Unity)
$\Rightarrow w = \sqrt[3]{-1}$
$= -1$
plus other two evenly spaced.



N.B.
these two are conjugates (reflection in real axis).

Question 4 **(7 marks)**

Consider the following system of equations, where k is a real constant.

$$x+2y+z=3$$
$$2x-y-3z=k$$
$$x+3y+kz=6$$

(a) Solve the system of equations when $k=1$. (3 marks)

$x+2y+z=3$ --- (i)
$2x-y-3z=1$ --- (ii)
$x+3y+z=6$ --- (iii)

(iii) − (ii) $\underline{y=3}$ ✓

$\Rightarrow x+z=-3$ and $2x-3z=4$

$\Rightarrow z=-3-x$ $\quad 2x-3(-3-x)=4$

$\Rightarrow$ $\quad \Rightarrow 2x+9+3x=4$

$\Rightarrow 5x=-5$

$z=-3-(-1)$ $\quad \therefore \underline{x=-1}$ ✓
$\underline{=-2}$

$\therefore (-1, 3, -2)$ # ✓

(b) Show that no value of k exists for the system of equations to represent three planes intersecting in a single straight line. (4 marks)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & -3 & k \\ 1 & 3 & k & 6 \end{array}\right] \begin{array}{l} \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 6-k \\ 0 & 1 & k-1 & 3 \end{array}\right] \begin{array}{l} \text{(i)} \\ \text{(ii)} \leftarrow 2\times\text{(i)}-\text{(ii)} \\ \text{(iii)} \leftarrow \text{(iii)}-\text{(i)} \end{array}$$ ✓

N.B. This is a bit like proof by contradiction!

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 6-k \\ 0 & 0 & 5k-10 & k+9 \end{array}\right] \begin{array}{l} \text{(i)} \\ \text{(ii)} \\ \text{(iii)} \leftarrow 5\times\text{(iii)}-\text{(ii)} \end{array}$$ ✓

Now for infinite solutions as requested: but $2 \neq -9$
$5k-10=0 \Rightarrow \underline{k=2}$ and $k+9=0 \Rightarrow \underline{k=-9}$ ✓ $\therefore$ No value of k ✓

QED.

Question 5 **(8 marks)**

(a) Determine the vector equation of the plane that contains the points A(1, -1, 2), B(2, 1, 0) and C(3, -1, 1). (4 marks)

$$AB = \begin{pmatrix}1\\2\\-2\end{pmatrix} \qquad AC = \begin{pmatrix}2\\0\\-1\end{pmatrix} \quad \checkmark$$

$$AC \times AB = \begin{pmatrix}1\\2\\-2\end{pmatrix} \times \begin{pmatrix}2\\0\\-1\end{pmatrix}$$

$$= \begin{pmatrix}2\\3\\4\end{pmatrix} \quad \checkmark$$ Page 3 of Formula Sheet.

$$= \underset{\sim}{n}$$

Now: $\underset{\sim}{r} \cdot \underset{\sim}{n} = c$

$$\Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} \cdot \begin{pmatrix}2\\3\\4\end{pmatrix} \quad \checkmark$$

$$\Rightarrow \underline{\underset{\sim}{r} \cdot (2,3,4) = 7} \quad \checkmark$$

Alternative method:

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a}) + \mu(\underset{\sim}{c} - \underset{\sim}{a})$$

$$= \begin{pmatrix}1\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-2\end{pmatrix} + \mu\begin{pmatrix}2\\0\\-1\end{pmatrix}$$

OR a cyclic treatment of points A, B + C.

(b) Plane Π has equation $x + 2y - z = 3$. Line L is perpendicular to Π and passes through the point (1, -6, 4). Determine where line L intersects plane Π. (4 marks)

Π: $x + 2y - z = 3$

$$\Rightarrow \underset{\sim}{r} \cdot (1,2,-1) = 3 \quad \checkmark$$

L: $\underset{\sim}{r} = (1,-6,4) + \lambda(1,2,-1)$

$$= \begin{pmatrix}1+\lambda\\2\lambda-6\\4-\lambda\end{pmatrix} \quad \checkmark$$

Sub. for $\underset{\sim}{r}$

$$\begin{pmatrix}1+\lambda\\2\lambda-6\\4-\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix} = 3 \quad \checkmark$$

$$\Rightarrow 1 + \lambda + 4\lambda - 12 - 4 + \lambda = 3$$

$$\Rightarrow 6\lambda = 18$$

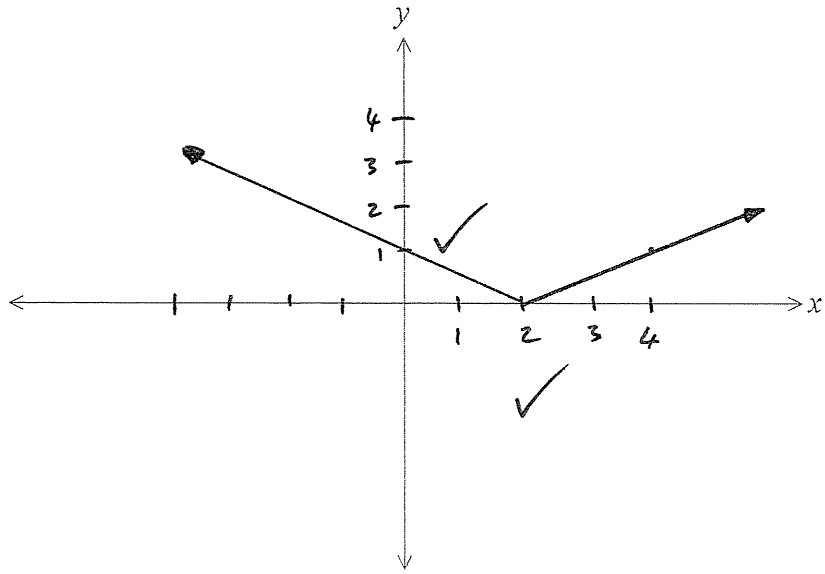
$$\therefore \underline{\lambda = 3}$$

$$\therefore \underset{\sim}{r} = \begin{pmatrix}4\\0\\1\end{pmatrix} \quad \checkmark$$

N.B. There are a number of alternative methods.

Question 6 **(7 marks)**

(a) Sketch the graph of $y = \frac{|x-2|}{2}$ on the axes below. (2 marks)



(b) Solve the equation $4|x-8| = 38 - x$. (3 marks)

$4(x-8) = 38 - x$ or $4(8-x) = 38 - x$

$\Rightarrow 4x - 32 = 38 - x$ $\Rightarrow 32 - 4x = 38 - x$

$\Rightarrow 5x = 70$ $\Rightarrow -3x = 6$

$\therefore \underline{\underline{x = 14}}$ $\therefore \underline{\underline{x = -2}}$

i.e. $\underline{\underline{x = -2 \text{ or } 14}}$

(c) Solve the inequality $\frac{1}{|x+2|} \leq 1$. (2 marks)

$x + 2 \geq 1$ or $-(x+2) \geq 1$

$\Rightarrow \underline{\underline{x \geq -1}}$ $\Rightarrow x + 2 \leq -1$

$\Rightarrow \underline{\underline{x \leq -3}}$

$\therefore \underline{\underline{x \leq -3 \text{ or } x \geq -1}}$

Question 7 **(10 marks)**

Particle A has position vector given by $\mathbf{r} = 3\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}$, where t is the time in seconds.

(a) Show that the path of the particle is circular. (2 marks)

You could argue its bleeding obvious, or show $\underset{\sim}{a} = -\underset{\sim}{r}$ $\Rightarrow$ circular motion

$\Rightarrow x = 3\cos t \quad , \quad y = 3\sin t$

$\Rightarrow \frac{x}{3} = \cos t \quad , \quad \frac{y}{3} = \sin t$

Now: $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2 t + \sin^2 t = 1$ ✓

$\Rightarrow \underline{\underline{x^2 + y^2 = 9}}$ ie. a Circle centre $\underline{\underline{(0,0)}}$; radius $= \sqrt{9} = \underline{\underline{3}}$ ✓

Particle B is stationary, with position vector $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

(b) Determine an expression for the distance between particles A and B in terms of t. (2 marks)

$|BA| = |OA - OB|$

$= \sqrt{(3\cos t - 3)^2 + (3\sin t - 4)^2 + (-5)^2}$

(c) Determine the position vector of particle A when it is (i) nearest and (ii) furthest from particle B. (6 marks)

Consider $|BA|$ or easier $|BA|^2$

✓

$\frac{d}{dt}\left(|BA|^2\right) = 2(3\cos t - 3)(-3\sin t) + 2(3\sin t - 4)(3\cos t)$

Let $0 = -\sin t(3\cos t - 3) + \cos t(3\sin t - 4)$

$\Rightarrow 0 = -\cancel{3\cos t}\sin t + 3\sin t + \cancel{3\cos t}\sin t - 4\cos t$

$\Rightarrow 0 = 3\sin t - 4\cos t$ ✓

$\Rightarrow \tan t = \frac{4}{3}$ ✓ $\therefore \sin t = \pm\frac{4}{5}, \cos t = \pm\frac{3}{5}$ ✓

and so

(i) nearest to B $\quad OA = \underline{\underline{\frac{9}{5}\underset{\sim}{i} + \frac{12}{5}\underset{\sim}{j}}}$ ✓

(ii) furthest to B $\quad OA = \underline{\underline{-\frac{9}{5}\underset{\sim}{i} - \frac{12}{5}\underset{\sim}{j}}}$ ✓

End of questions

TRINITY COLLEGE

Semester One Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3

Section Two: Calculator-assumed

If required by your examination administrator, please place your student identification label in this box

Student Number: In figures

In words SOLUTIONS

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Reading time before commencing work: ten minutes
Working time for section: one hundred minutes

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Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

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Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 8 **(5 marks)**

Consider the function $f(x) = x^2 - 4x$.

(a) Explain why it is necessary to restrict the natural domain of f in order that its inverse is also a function. (1 mark)

Because it is not a 1-1 function ✓, but rather a m-1 function. The inverse of a m-1 function would be 1-m, such relations are not functions.

(b) State a minimal restriction to the domain of f that includes $x = 3$, and then use this restriction to show that $f^{-1}(x) = 2 + \sqrt{x+4}$. (4 marks)

$f(x) = x(x-4)$

$f(2) = -4$ ie. Turning point $(2, -4)$ *

hence minimal restriction is $x \geq 2$ (includes $x=3$) ✓

Consider $y = x^2 - 4x$, $x \geq 2$

$\Rightarrow y + 4 = x^2 - 4x + 4$ Complete the square.

$\Rightarrow y + 4 = (x-2)^2$ ✓

$\Rightarrow \sqrt{y+4} = x - 2$

$\Rightarrow x = 2 + \sqrt{y+4}$ ✓

Interchange x + y

$y = 2 + \sqrt{x+4}$

$\Rightarrow f^{-1}(x) = 2 + \sqrt{x+4}$ ✓ QED. , $x \geq 2$

*Comment: May be tempted to think $x \geq -4$ but infact

Question 9 **(5 marks)**

(a) Let z be a non-zero complex number located in the complex plane. Describe the linear transformation(s) required to transform z to each of the following locations:

(i) $2z$. (1 mark)

Dilation of scale factor 2 about the origin. ✓

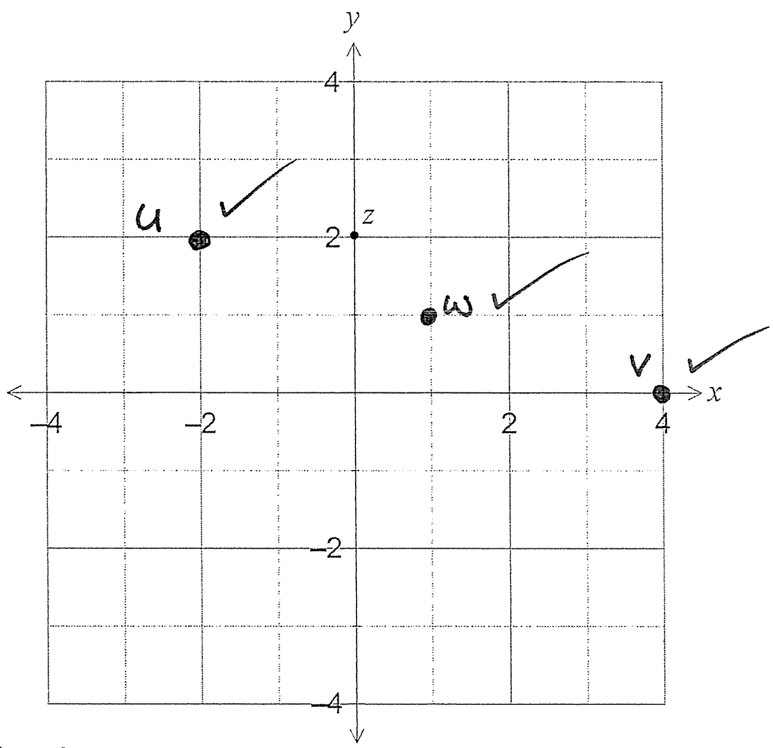
(ii) $i^3 z$. (1 mark)

Rotation of $-\frac{\pi}{2}$ about the origin.

Clockwise ✓

(b) Consider the complex number z shown in the Argand diagram below. Add to the diagram the location of u, v and w where $u=(1+i)z$, $v=z\cdot\bar{z}$ and $w=\sqrt{z}$. (3 marks)

$z = 2i$



Can use your Calculator!

$u = (1+i)2i$
$= 2i - 2$
$= -2 + 2i$

$v = 2i(-2i)$
$= -4i^2$
$= 4$

$w = \sqrt{2i}$
$= \sqrt{2}\sqrt{i}$
$= \sqrt{2}\,\text{cis}\frac{\pi}{4}$
$= 1 + i$

where $\text{cis}\frac{\pi}{2} = i$
$\Rightarrow \left(\text{cis}\frac{\pi}{2}\right)^{\frac{1}{2}} = i^{\frac{1}{2}}$
$\Rightarrow \text{cis}\frac{\pi}{4} = \sqrt{i}$

Question 10 **(8 marks)**

Two functions are given by $f(x) = 2\sqrt{x+1}$ and $g(x) = x^2 - 2x$.

(a) Determine $g \circ f(x)$ and state the domain and range of this composite function. (3 marks)

$$g(f(x)) = \left(2\sqrt{x+1}\right)^2 - 2\left(2\sqrt{x+1}\right)$$
$$= 4(x+1) - 4\sqrt{x+1} \checkmark$$

Domain: $x \geq -1$, $x \in \mathbb{R}$ ✓

Range: $y \geq -1$, $y \in \mathbb{R}$ ✓ by Calculator, otherwise need Calculus: $\left.\frac{d}{dx} g(f(x))\right|_{x=-\frac{3}{4}} = 0$; $g\left(f\left(-\frac{3}{4}\right)\right) = \underline{\underline{-1}}$

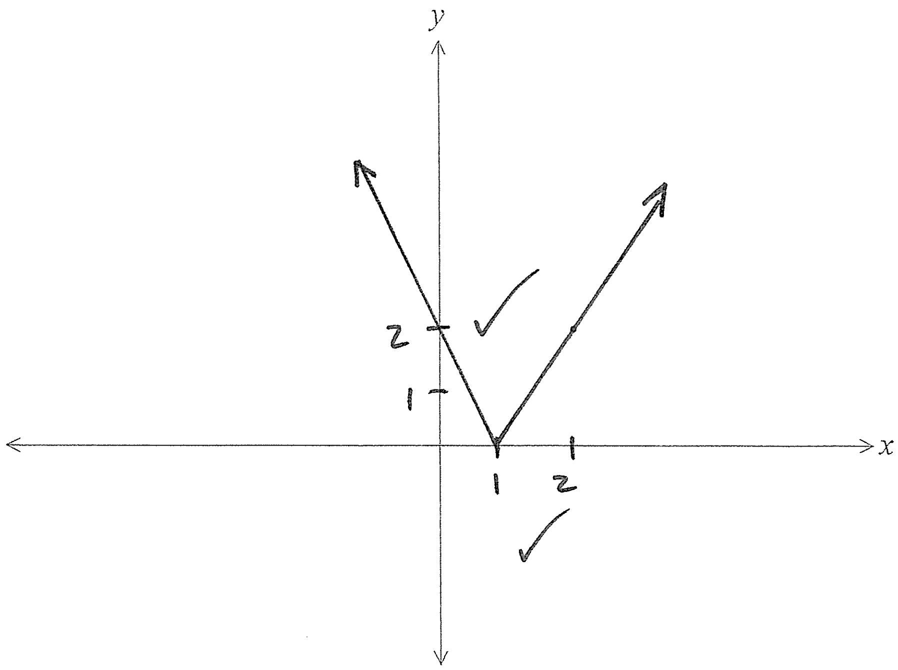
(b) Show that the composite function $f \circ g(x)$ is defined for $x \in \mathbb{R}$. (3 marks)

$$f(g(x)) = 2\sqrt{x^2 - 2x + 1} \checkmark$$
$$= 2\sqrt{(x-1)^2} \checkmark$$
$$= 2|x-1|$$
$$= \begin{cases} 2 - 2x, & x < 1 \\ 2x - 2, & x \geq 1 \end{cases}$$

✓ i.e. Defined for $x \in \mathbb{R}$

QED.

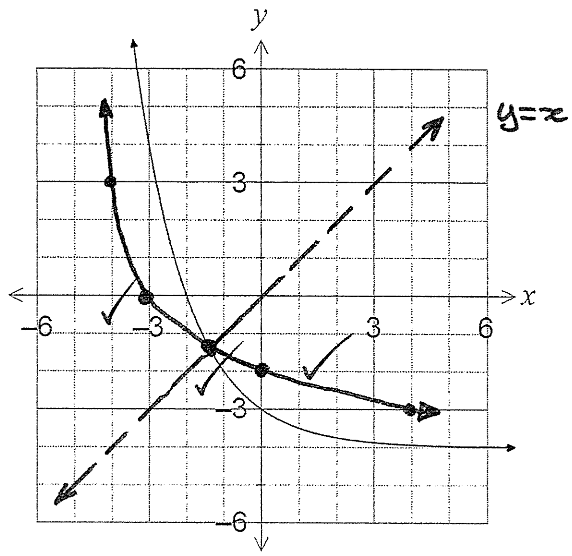
(c) Sketch the graph of $y = f \circ g(x)$ on the axes below. (2 marks)



✓ ✓

Question 11 **(12 marks)**

(a) The graph of $y = f(x)$ is shown below.



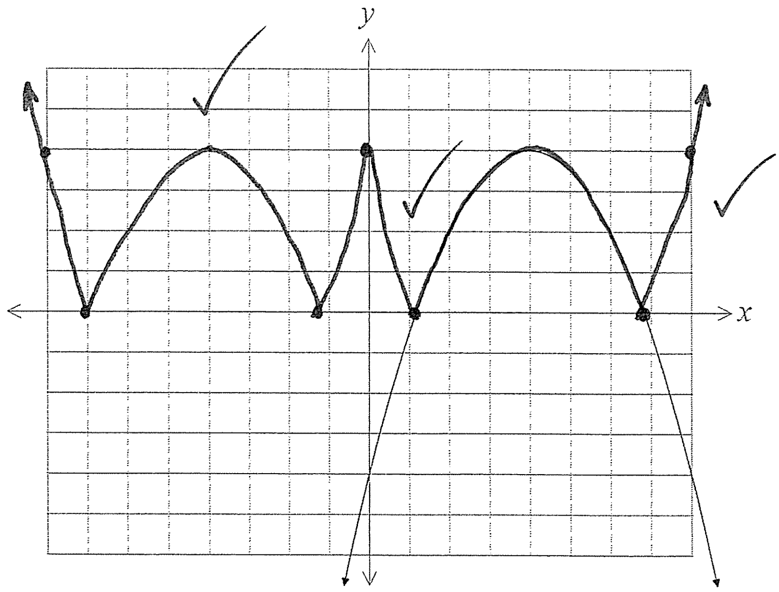
(i) What feature of the graph suggests that the inverse of f is a function? (1 mark)

The graph is a function (Vertical Line Test)
in particular is a 1-1 function (Horizontal Line Test)

(ii) On the same axes, sketch the graph of the inverse of f, $y = f^{-1}(x)$. (3 marks)

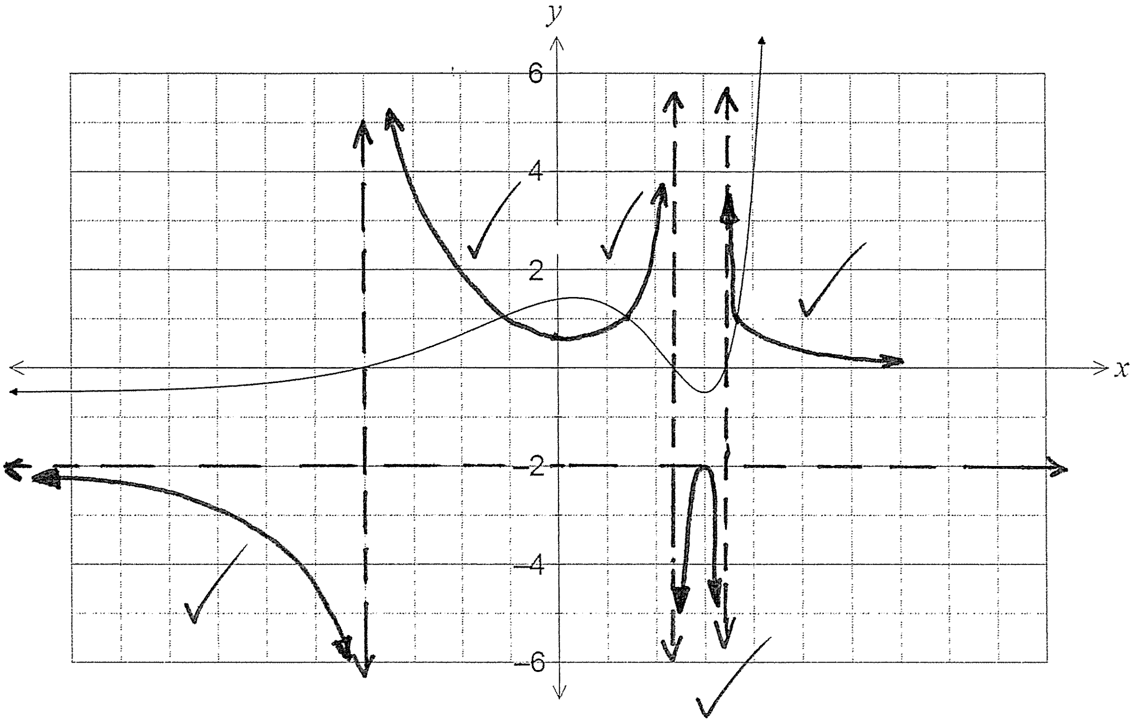
See above: reflection in y=x ; accuracy.

(b) The graph of $y = g(x)$ is shown below.
On the same axes, sketch the graph of $y = |g(|x|)|$. (3 marks)



(c) The graph of $y = h(x)$ is shown below. As $x \to -\infty$, $h(x) \to -0.5$. On the same axes, sketch the graph of $y = \dfrac{1}{h(x)}$, clearly indicating all vertical and horizontal asymptotes.

(5 marks)



Given $h(x) \to -\frac{1}{2}$ as $x \to -\infty$

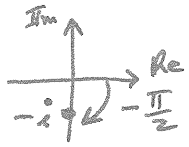
Reciprocals

$\frac{1}{h(x)} \to -2$ as $x \to -\infty$ Horizontal asymptote

Where ever $h(x) = 0$, $\frac{1}{h(x)}$ has a Vertical asymptote.

Question 12 **(8 marks)**

(a) Determine all roots of the equation $z^6 + 8i = 0$, expressing them in exact polar form $r cis\theta$ where $r > 0$ and $-\pi < \theta \le \pi$. (5 marks)



$$z^6 = -8i$$

$$= 8\,\text{cis}\left(\frac{-\pi}{2}\right) \checkmark$$

$$\Rightarrow z = \left(8\,\text{cis}\left(\frac{-\pi}{2}\right)\right)^{\frac{1}{6}} \checkmark$$

De Moivre

$$= \sqrt[6]{8}\ \text{cis}\left(\frac{-\frac{\pi}{2} + 2\pi k}{6}\right), \quad k = -2, -1, 0, 1, 2, 3$$

$$= (2^3)^{\frac{1}{6}}\ \text{cis}\left(\frac{-\pi + 4\pi k}{12}\right)$$

$$= \sqrt{2}\ \text{cis}\left(-\frac{\pi}{12} + \frac{\pi k}{3}\right) \checkmark$$

$\therefore z_0 = \sqrt{2}\,\text{cis}\left(\frac{-\pi}{12}\right)$ $\qquad z_{-1} = \sqrt{2}\,\text{cis}\left(-\frac{5\pi}{12}\right)$

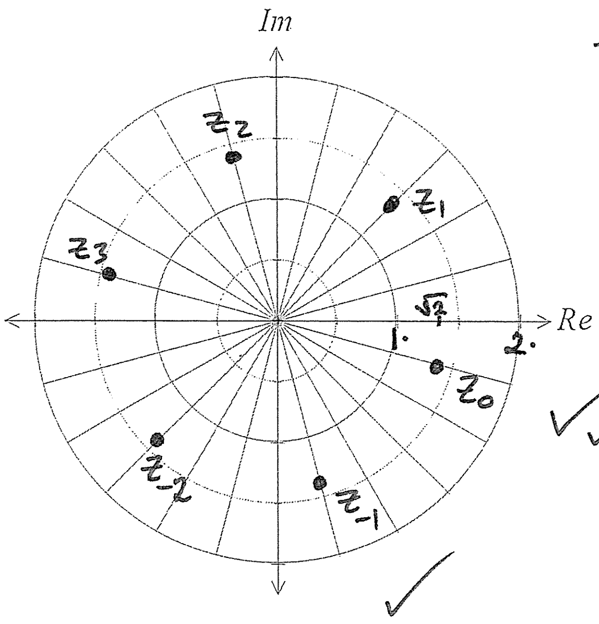
$z_1 = \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$ $\qquad z_{-2} = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

$z_2 = \sqrt{2}\,\text{cis}\left(\frac{7\pi}{12}\right)$

$z_3 = \sqrt{2}\,\text{cis}\left(\frac{11\pi}{12}\right)$ ✓ ✓

(b) Show all solutions of the equation in part (a) on the Argand diagram below. (3 marks)

z_k



✓✓

✓

Question 13 **(7 marks)**

Two small bodies, A and B, simultaneously leave their initial positions of $\mathbf{i}+4\mathbf{j}-25\mathbf{k}$ and $16\mathbf{i}+\mathbf{j}-2\mathbf{k}$, and move with constant velocities of $4\mathbf{i}+\mathbf{j}+5\mathbf{k}$ and $-\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ respectively.

(a) Determine whether the paths of the bodies cross or if the bodies meet. (4 marks)

$$\underset{\sim}{r}_A = \begin{pmatrix} 1+4t_1 \\ 4+t_1 \\ -25+5t_1 \end{pmatrix} \qquad \underset{\sim}{r}_B = \begin{pmatrix} 16-t_2 \\ 1+2t_2 \\ -2-3t_2 \end{pmatrix} \quad \checkmark$$

Let
$$\left.\begin{aligned} 1+4t_1 &= 16-t_2 \\ 4+t_1 &= 1+2t_2 \\ -25+5t_1 &= -2-3t_2 \end{aligned}\right|$$
ClassPad

No solution for t_1, t_2 ✓

ie. System has no solution

$\Rightarrow$ Paths do not cross ✓

$\Rightarrow$ Do not meet. ✓

(b) At the same time, a third small body, C, leaves its initial position, passes through the origin and crosses the path of body A. If C moves with a steady velocity of $5a\mathbf{i}+5\mathbf{j}+a\mathbf{k}$, determine the value of the constant a. (3 marks)

$$\underset{\sim}{r}_C = \begin{pmatrix} 5at \\ 5t \\ at \end{pmatrix} \quad \checkmark$$

Set $\underset{\sim}{r}_A = \underset{\sim}{r}_C$

$$\left.\begin{aligned} 1+4t_1 &= 5at_2 \\ 4+t_1 &= 5t_2 \\ -25+5t_1 &= at_2 \end{aligned}\right| \quad t_1, t_2, a \quad \checkmark$$

$\therefore \underline{\underline{a = 2.5}};\ t_1 = 6;\ t_2 = 2$

✓

Question 14 **(9 marks)**

The function f is defined by $f(x) = \dfrac{x^2 - 1}{x^2 - 3x + 2}$.

(a) Determine the natural domain and range of $f(x)$. (4 marks)

$$f(x) = \frac{(x+1)\cancel{(x-1)}}{(x-2)\cancel{(x-1)}}$$

$$= \frac{1 + \frac{1}{x}}{1 - \frac{2}{x}}$$

$$= 1 \text{ as } x \to \pm\infty$$

Horizont Asymptote

Vertical Asymptote

$x \neq 1,\ x \neq 2,\ x \in \mathbb{R}$ ✓✓

Domain. Hole at $(1, -2)$

Hole at $(1, -2)$

$\therefore$ Range: ✓✓

$y \neq -2,\ y \neq 1,\ y \in \mathbb{R}$

Alternatively:

$$f(x) = 1 + \frac{3}{x-2}$$

$$= 1 \text{ as } x \to \pm\infty$$

$$\begin{array}{r|l} & 1 \\ x-2 & x+1 \\ & x-2 \\ \hline & 3 \end{array}$$

(b) Show that the function has no stationary points. (2 marks)

$$f'(x) = -3(x-2)^{-2} \cdot 1$$

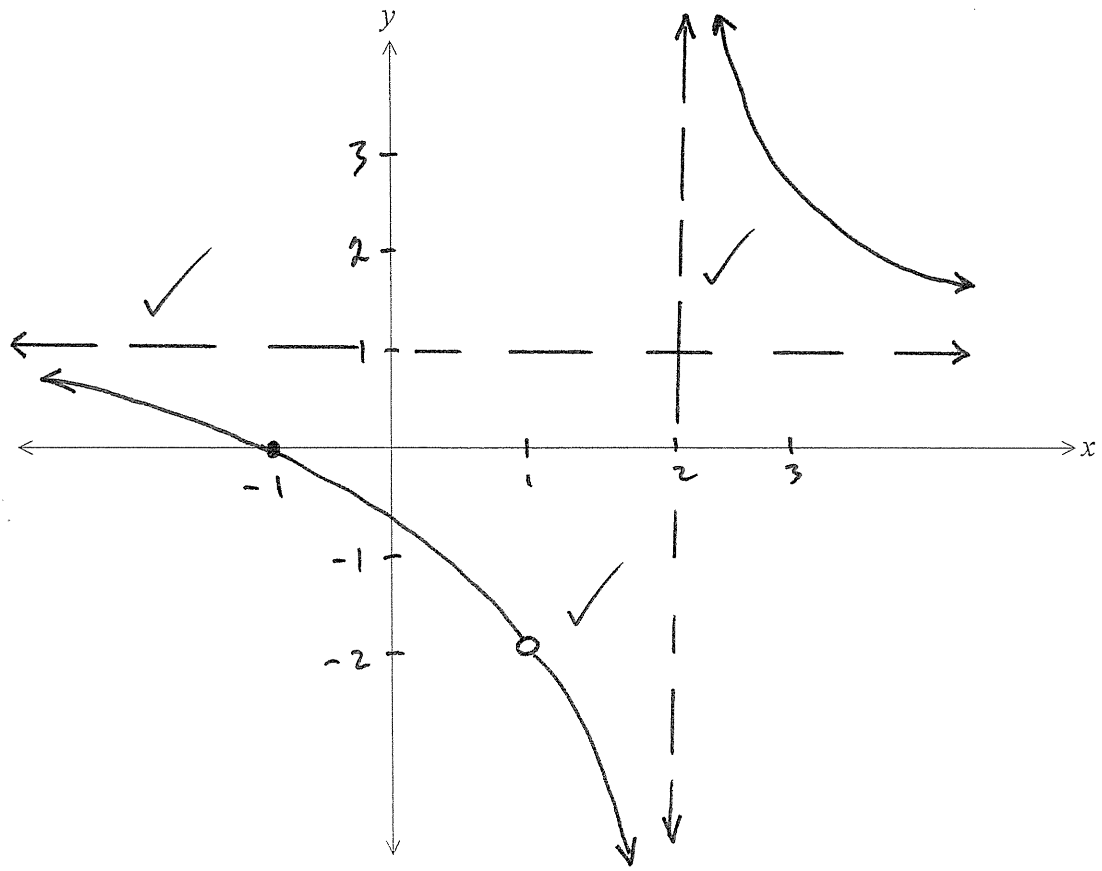
$$= \frac{-3}{(x-2)^2} \checkmark$$

and $-3 \neq 0$

$\therefore$ $f'(x) \neq 0$ ✓

i.e. no stationary points.

(c) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)



Question 15 this question is testing the arguement and modulus properties of $\mathbb{C}$. **(8 marks)**

Given the two complex numbers $w = r(\cos\theta + i\sin\theta)$ and $z = s(\cos\phi + i\sin\phi)$, determine the following in terms of the non-zero constants r, s, θ and ϕ:

(a) $\arg(\overline{z})$. (1 mark)

$z = r\,\text{cis}\,\phi$

$\Rightarrow \overline{z} = r\,\text{cis}(-\phi)$

$\Rightarrow \arg(\overline{z}) = \underline{\underline{-\phi}}$ ✓

(b) $\left|\dfrac{i}{w^2}\right|$. (2 marks)

Modulus

$$\frac{i}{w^2} = \frac{\text{cis}\left(\frac{\pi}{2}\right)}{r^2\,\text{cis}\,2\theta} \quad ✓$$

$$= \frac{1}{r^2}\left(\text{cis}\left(\frac{\pi}{2} - 2\theta\right)\right)$$

$$\Rightarrow \left|\frac{i}{w^2}\right| = \underline{\underline{\frac{1}{r^2}}} \quad ✓$$

(c) $|(1-i)wz|$. (2 marks)

$= \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right) \cdot r\,\text{cis}\,\theta \cdot s\,\text{cis}\,\phi$ ✓

$= \sqrt{2}\,rs\,\text{cis}\left(\theta + \phi - \frac{\pi}{4}\right)$

$\Rightarrow |(1-i)wz| = \underline{\underline{\sqrt{2}\,rs}}$ ✓

(d) $\arg\left(-\dfrac{z}{iw}\right)$. (3 marks)

$$-\frac{z}{iw} = \frac{\text{cis}(\pi) \cdot s\,\text{cis}\,\phi}{\text{cis}\left(\frac{\pi}{2}\right) r\,\text{cis}\,\theta} \quad ✓$$

$$= \frac{s}{r}\,\text{cis}\left(\pi + \phi - \frac{\pi}{2} - \theta\right) \quad ✓$$

$$\Rightarrow \arg\left(\frac{-z}{iw}\right) = \underline{\underline{\frac{\pi}{2} + \phi - \theta}} \quad ✓$$

Question 16 **(7 marks)**

Consider the three vectors $\mathbf{a} = \langle 2,1,-3 \rangle$, $\mathbf{b} = \langle -3,5,-2 \rangle$ and $\mathbf{c} = \langle 2,-4,1 \rangle$.

(a) Prove that the three vectors do not lie in the same plane. (4 marks)

ie Free vectors (not position vectors, a.k.a. points)

For vectors to lie in the same plane, then a vector perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$ (namely $\underset{\sim}{a} \times \underset{\sim}{b}$) will ✓ also be perpendicular to $\underset{\sim}{c}$ (namely $(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c} = 0$) ✓

$$\left(\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix} \right) \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 \\ 13 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \checkmark$$

$$= \underline{\underline{-13}} \neq 0 \quad \therefore$$ ✓

The vectors cannot lie in the same plane. Q.E.D.

If treated as "three points and a line"

$\underset{\sim}{b} - \underset{\sim}{a} = \lambda(\underset{\sim}{c} - \underset{\sim}{a})$

$\Rightarrow \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ -5 \\ 4 \end{pmatrix}$

has no solution for λ

$\therefore$ Do not lie on the same <u>line</u>.

(b) Determine the value(s) of the constant a if the vector $\langle a^2, a, a-3 \rangle$ lies in the same plane as vectors $\mathbf{a}$ and $\mathbf{b}$. (3 marks)

Note: $\begin{pmatrix} 13 \\ 13 \\ 13 \end{pmatrix} = 13 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Consider $(1, 1, 1) \cdot (a^2, a, a-3) = 0$ ✓

$\Rightarrow a^2 + a + a - 3 = 0$ ✓

$\Rightarrow a^2 + 2a - 3 = 0$

$\Rightarrow (a+3)(a-1) = 0$

$\therefore \underline{\underline{a = -3 \text{ or } a = 1}}$ ✓

$\underset{\sim}{b} - \underset{\sim}{a} = \lambda \left(\begin{pmatrix} a^2 \\ a \\ a-3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \right)$

$\begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} a^2 - 2 \\ a - 1 \\ a \end{pmatrix}$

For whatever λ there is no value for constant 'a'.

Question 17 **(9 marks)**

Let the complex number $z = \cos\theta + i\sin\theta$.

(a) Show that $\frac{1}{z} = \cos\theta - i\sin\theta$. (2 marks)

Take L.H.S. $\frac{1}{z} = z^{-1}$

$= (\text{cis}\,\theta)^{-1}$ ✓

$= \text{cis}(-\theta)$ De Moivre

$= \cos\theta - i\sin\theta$

$=$ R.H.S. Q.E.D. ✓

(b) Show that $z^3 - \frac{1}{z^3} = 2i\sin 3\theta$. (2 marks)

Take L.H.S. $z^3 - \frac{1}{z^3} = z^3 - z^{-3}$

$= \text{cis}\,3\theta - \text{cis}(-3\theta)$ ✓

$= \cos 3\theta + i\sin 3\theta - (\cos(-3\theta) + i\sin(-3\theta))$

$= \cancel{\cos 3\theta} + i\sin 3\theta - \cancel{\cos(3\theta)} + i\sin(3\theta)$

$= 2i\sin 3\theta$ ✓

$=$ R.H.S. Q.E.D.

(c) Determine $\text{Im}\left(z^3 - \frac{1}{z^3}\right)$ in terms of $\sin\theta$ and $\cos\theta$. (3 marks)

$= 2\sin 3\theta$ from (b) ✓

$= 2\sin(2\theta + \theta)$

$= 2(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta)$ ✓

$= 2(2\sin\theta\cos\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta)$

$= 4\sin\theta\cos^2\theta + 2(\cos^2\theta\sin\theta - \sin^3\theta)$

$= \underline{\underline{6\cos^2\theta\sin\theta - 2\sin^3\theta}}$ ✓

<u>Alternatively</u> expand z^3 and z^{-3} and collect imaginary parts.

(d) Express $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$. (2 marks)

$2\sin 3\theta = 6\cos^2\sin\theta - 2\sin^3\theta$

$\Rightarrow 2\sin 3\theta = 6(1-\sin^2\theta)\sin\theta - 2\sin^3\theta$ ✓

$\Rightarrow \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\therefore \sin^3\theta = \frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4}$ ✓

as required.

Question 18 **(13 marks)**

The velocity vector of a particle at time t seconds is $\mathbf{v}(t) = 3\mathbf{i} - \dfrac{3}{t^2}\mathbf{j}$, for $t \geq 1$. When $t = 1$, the particle has position vector $2\mathbf{j}$.

(a) Calculate the exact speed of the particle when $t = 2$. (2 marks)

$$\underset{\sim}{v}(2) = 3\underset{\sim}{i} - \frac{3}{4}\underset{\sim}{j}$$ ✓

$$\Rightarrow |\underset{\sim}{v}(2)| = \frac{3\sqrt{17}}{4}$$ ✓

(b) Determine the acceleration vector of the particle and comment on its direction. (2 marks)

$$\underset{\sim}{a}(t) = 6t^{-3}\underset{\sim}{j}$$

$$= \frac{6}{t^3}\underset{\sim}{j}$$ ✓

Acceleration has no $\underset{\sim}{i}$ component and so always ✓ acts parallel to y-axis. (upwards)

(c) Determine the position vector of the particle for $t \geq 1$. (2 marks)

$$\underset{\sim}{r}(t) = (3t + c_1)\underset{\sim}{i} + \left(\frac{3}{t} + c_2\right)\underset{\sim}{j}$$ ✓

$$\underset{\sim}{r}(1) = (3 + c_1)\underset{\sim}{i} + (3 + c_2)\underset{\sim}{j} = 2\underset{\sim}{j} \quad \text{given.}$$

Equate component parts

$3 + c_1 = 0$ $\quad$ $3 + c_2 = 2$

$\therefore c_1 = -3$ $\quad$ $c_2 = -1$

i.e. $\underset{\sim}{r}(t) = (3t - 3)\underset{\sim}{i} + \left(\frac{3}{t} - 1\right)\underset{\sim}{j}$ ✓

(d) Derive the Cartesian equation of the path of the particle in the form $y = f(x)$. (2 marks)

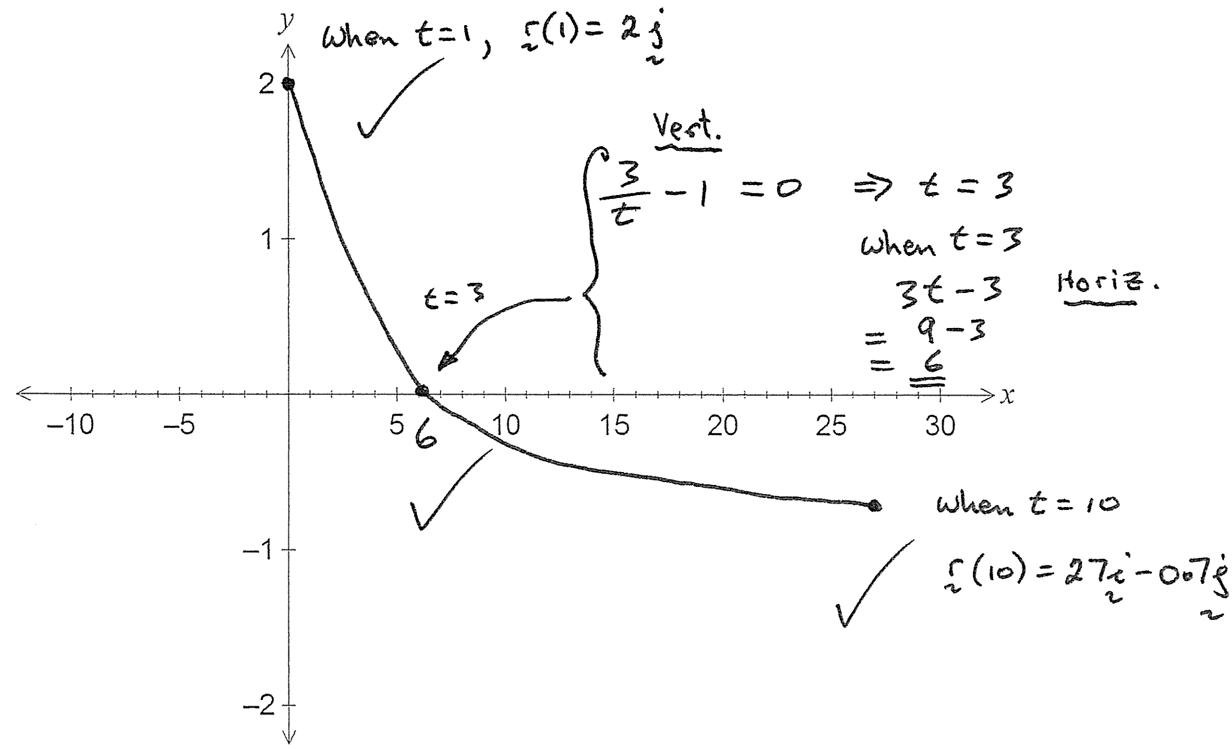
$x = 3t - 3$ and $y = \frac{3}{t} - 1$

$$\Rightarrow t = \frac{x+3}{3}$$ ✓

$$\Rightarrow y = \frac{3}{\frac{x+3}{3}} - 1$$

$$\therefore y = \frac{9}{x+3} - 1$$ as requested. ✓

(e) On the axes below, sketch the path taken by the particle for $1 \le t \le 10$, clearly indicating the position of the particle at the start and end of this interval. (3 marks)



(f) Determine the length of the path travelled by the particle between $t = 1$ and $t = 10$. (2 marks)

Length of Path (Distance Travelled) $= \int_1^{10} |\underset{\sim}{v}(t)|\, dt$

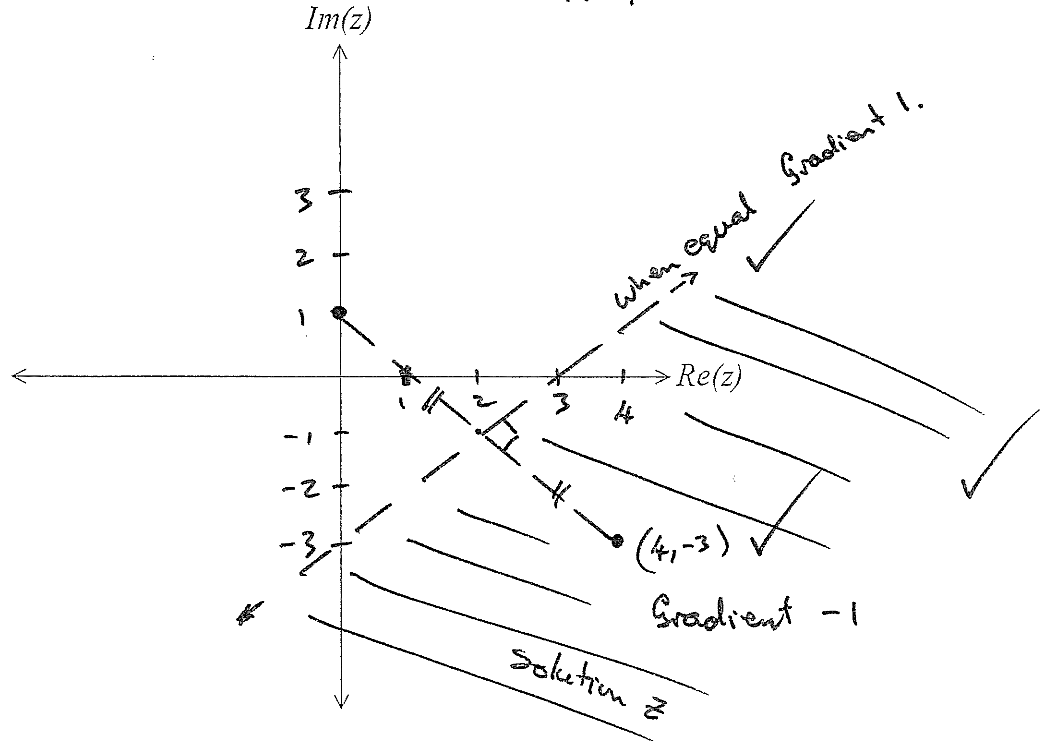
$$= \int_1^{10} \sqrt{(3)^2 + \left(-\frac{3}{t^2}\right)^2}\, dt$$

$= \underline{27.46}$ units (2d.p.)

Question 19 **(7 marks)**

(a) Shade the region satisfying the complex inequality $|z-i| > |z-4+3i|$ on the Argand diagram below. (3 marks)

$|z-(0+i)| > |z-(4-3i)|$

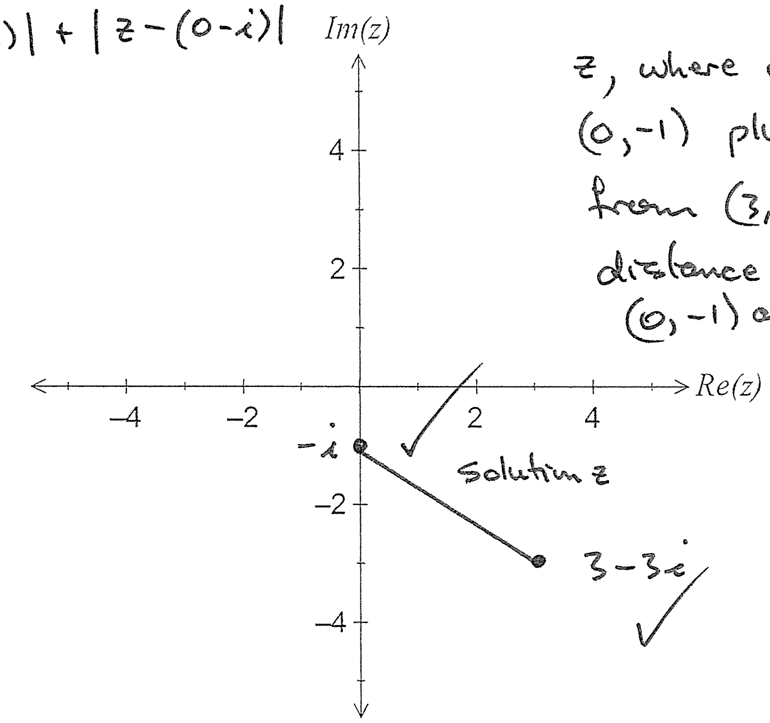


(b) Consider the two complex numbers given by $u = 3-3i$ and $v = -i$. Sketch each of the following sets of points in the complex plane.

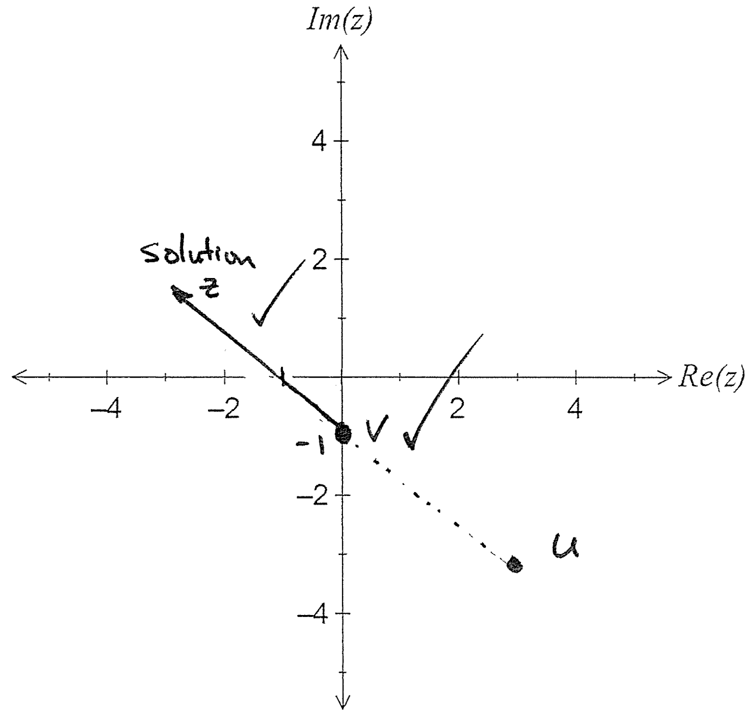
(i) $|z-u|+|z-v| = |u-v|$. (2 marks)

$|z-(3-3i)| + |z-(0-i)|$

z, where distance from (0,-1) plus distance from (3,-3) equals distance between (0,-1) and (3,-3).



(ii) $|z-v|+|u-v|=|z-u|$. (2 marks)



ie. Distance of z from v plus

distance of u from v

equals distance of z from u.